Subgraph characterization of Red/Blue-Split Graph and Kőnig Egerváry Graphs

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Abstract

Kőnig-Egerváry graphs (KEGs) are the graphs whose maximum size of a matching is equal to the minimum size of a vertex cover. We give an excluded subgraph characterization of KEGs. We show that KEGs are a special case of a more general class of graph: *Red/Blue-split* graphs, and give an excluded subgraph characterization of Red/Blue-split graphs. We show several consequences of this result including theorems of Deming-Sterboul, Lovász, and Földes-Hammer. A refined result of Schrijver on the integral solution of certain systems of linear inequalities is also given through the result on the weighted version of Red/Blue-split graphs.

1 Introduction

Given an undirected graph G, let $\nu(G)$ be the maximum size of a matching, and $\tau(G)$ be the minimum size of a vertex cover, which is the minimum size of a node set covering all the edges. Finding τ and ν is a traditional problem in combinatorial optimization. The first work on this problem is due to Dénes Kőnig [8] and Jenő Egerváry [3] in 1931. They proved that in every bipartite graph G, $\tau(G) = \nu(G)$. Their work has a fundamental influence on Combinatorial Optimization. There have been many generalizations for this problem during the last fifty years. While the maximum matching problem can be extended to general graphs, the dual problem: minimum vertex cover is $\mathcal{NP}$-hard. Due to Kőnig and Egerváry in bipartite graphs a minimum vertex cover can be found in polynomial time. Extending this result, one can ask what are the graphs whose maximum size of a matching is equal to the minimum size of the vertex cover? In the literature these graphs are called Kőnig-Egerváry Graphs (KEGs for short) or graphs with the Kőnig property. It's clear that in this class of graphs the minimum vertex cover is in $\mathcal{P}$. KEGs have been studied in [2], [16], [13], [12], [10], [9], [6].

Our result: In this paper we give a *subgraphs excluded* characterization for Kőnig-Egerváry graphs: We show that given a graph and a maximum matching, the graph is KEG iff it doesn't contain one of the forbidden subgraphs in Figure 1. In this Figure the dashed edges are alternating paths starting and ending with matching edges, the thick edges correspond to alternating paths starting and ending with non-matching edges. We would like to note that there are interesting relations between a KEG and the vertex packing polyhedra. In the full paper we show that some of the configurations of Figure 1 are facet producing subgraphs.

We show that a KEG is a special case of a more general model class of graph: *Red/Blue-split* graphs, (R/B-split for short). These are simple graphs whose edges are colored by red, blue or both and the node set can be partitioned into a red and a blue stable set. (Red or blue stable set is a stable set in the graph made of red or blue edges respectively). The characterization of KEGs is shown by the characterization of Red/Blue-split graphs. The weighted version of R/B-split graphs is also considered, detailed proofs and connections with Schrijver's theorem are discussed in the full version of this paper.

R/B-split graphs were first studied by Gavril [**?**], who observed that R/B-split graphs yield a common generalization of KEGs and *split graphs*. Split graphs introduced in [5] are the graphs whose node set can be partitioned into a stable set and a clique. It's clear that if we color the edges of a graph red and its complement blue then the graph is split if and only if the corresponding colored graph is R/B-split. On the other hand, in a graph containing a perfect matching, if we color the matching edges red, and the others blue the graph is a KEG if and only if the colored one is R/B-split. For a graph not containing a perfect matching, with a technical operation we can reduce the problem to the case if the graph has a perfect matching. (See section 4).

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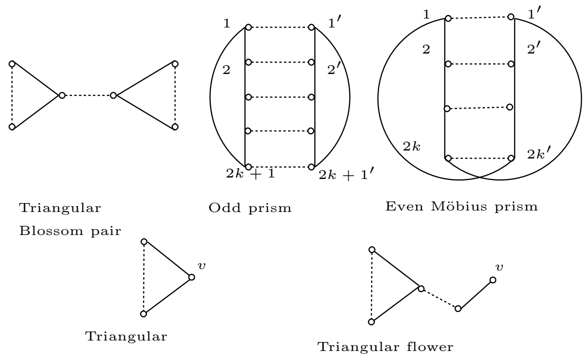

Figure 1: Forbidden subgraphs, v is not adjacent to matching edges (dashed edges).

Related work: There is a large literature characterizing various graph properties via excluded minors or excluded subgraphs. The first of these is Kuratowski's Theorem [11] that states that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$. Wagner [17] has shown that a graph is planar if and only if it has no minor isomorphic to K_5 or $K_{3,3}$. In the 80's and 90's Robertson and Seymour [14] in a sequence of papers have shown that all minor closed graph properties can be characterized by excluding a finite set of minors. For many interesting graph properties small classes of excluded minors have been found. The strong perfect graph theorem, which was proved recently by M. Chudnovsky, N. Robertson, R. Thomas and P.Seymour [1], also belongs to this topic.

Deming [2] and Sterboul [16] independently gave a characterization of KEGs in terms of some complicated forbidden configurations, however their characterization is not in terms of excluded subgraphs. Lovász [13] characterized KEGs using the theory of *matching-cover graphs*. His characterization requires that the graph G does not contain certain subgraphs for a *specific* perfect matching in G. We give the simple excluded subgraph characterization mentioned above. Our work is partially based on an earlier version [9].

Schrijver [15] used the Fourier-Motzkin elimination of variables to solve and characterize the existence of an integral solution of the inequalities system $Ax \leq b$, where b is an integral vector and A is an integral matrix satisfying $\sum_j |a_{ij}| \leq 2$. The weighted version of R/B-split graphs gives a refined result on this problem.

Földes and Hammer [5] introduced an interesting class of graphs: Split graph. These are the graphs whose node set can be partitioned into a stable set and a clique. They proved that a graph is a split graph iff it contains no induced subgraphs isomorphic to $2K_2, C_4$ or C_5. Our model of R/B-split graphs also resolves this characterization.

In [4] U. Faigle, B. Fuchs, B. Peis look at the problem of maximizing the node set such that the induced graph is R/B-split. They proved that the problem is hard even for comparability graphs and there can be no algorithms with the approximation ratio better than $\frac{31}{32}$ unless $\mathcal{P} = \mathcal{NP}$. One can also ask what is the number of edges (both red and blue) that can be deleted to obtain an R/B-split graph. We remark that minimum vertex cover can also be asked in the form of this question. We hope further investigations on R/B-split graphs can give useful information about both lower and upper bound for some $\mathcal{NP}$-hard problems especially the minimum vertex cover.

Problem definition: Let $G = (V, E)$ denote a graph with nodes V and undirected edges E. In the present paper, we consider graphs $G = (V, E)$, whose edge set E consists of red and blue edges, say $E = R \cup B$. R and B are not necessarily disjoint. We moreover assume R and B to have neither loops nor parallel edges. We call $G = (V, R \cup B)$ *Red-Blue* graph, and $G_R = (V, R)$, $G_B = (V, B)$ *red* respectively *blue* graphs. We are interested in covering the node set V of such a graph G by a stable set in the blue graph, and a stable set in the red graph. (A subset is stable in a graph if there isn't any pair of nodes in the subset connected by an edge in the graph.) Suppose a Red-Blue graph has the property that the whole vertex set can be covered by a red and a blue stable set. Then we call the graph *Red/Blue-Split graph* (or an *R/B-Split graph* for short) as the node set can be split into a red and a blue stable set.

The paper is organized as follows: In section 2, we give a simple characterization of R/B-split graph. The forbidden configurations are not simple subgraphs. This characterization can be considered as the characterization of Deming and Sterboul. In section 3, we characterize R/B-split graphs by forbidden subgraphs. Using this we prove the characterizations of KEGs in section 4. In section 5 we show how theorems of Deming-Sterboul, Lovász and Földes-Hammer can be derived from our result and a theorem of weighted R/B-split graphs is claimed. Section 6 discusses some open problems.

2 A characterization of R/B-Split Graphs by odd cycles

Deciding whether or not G is an R/B-split graph can be reduced to a 2–Satisfiability Problem and vice versa [18]. In this section we give a simple characterization of R/B-split graphs, which is in fact the generalization of the characterization of Deming and Sterboul on KEGs. We are going to use this characterization to prove our main theorem in the next section.

First we give some notation: A walk with a sequence of colored alternating edges in a Red-Blue graph is called an *alternating walk*. As we consider only alternating walks in this paper, we simply write *walks* instead of alternating walks most of the time. Note that a walk might traverse nodes and edges more than once. *Size* of a walk is the number of edges on the walk. A walk is a *cycle* if the start- and endnode are identical. The start- resp. endnode of a cycle is said to be the *base* of the cycle. If a cycle's length is odd, then the starting and ending edges are of the same color. In this case we talk about *red* or *blue odd cycle* depending on that color. Note that G is an R/B-split graph if and only if the nodes can be colored red and blue in such a way that the red nodes form a red stable set and the blue nodes form a blue stable set. We denote such a coloring a *feasible coloring of G*.

THEOREM 2.1. *G is an R/B-split graph if and only if there exist no two different colored odd cycles with the same base.*

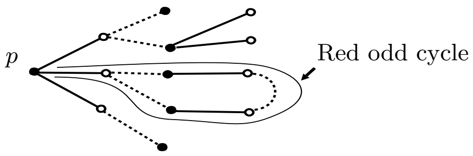

Figure 2: An alternating tree

Proof. Assume that we have a feasible coloring. Consider a walk starting with a red edge. If the starting node is colored red then the second node has to be colored blue, and thus the third node is red and so on. That is, all the nodes on the walk need to be colored alternately. Now if there is a red odd cycle with base v then v needs to be colored blue in any feasible coloring, and if there is a red and blue odd cycle with the same base v, then we find an evidence showing that there are no feasible colorings.

To prove sufficiency, pick an arbitrary node p, assume that there are no red odd cycles with base p. Color p red and color all the nodes that can be reached from p on a alternating path starting with a red edge with the proper color. This can be done by building an *alternating breath first search* tree from the root p. Because there are no red odd cycles on base p, there won't be two red (blue) nodes connected by a red resp. blue edge. In case not every node can be reached from p, pick an uncolored node and apply the above algorithm and repeat this until every node is colored. By picking an uncolored node and coloring it and all the nodes reachable from that with proper color, we can never hurt the coloring we already obtained. Thus, at the end we get a feasible coloring.

We just gave a simple characterization of R/B-split graphs. However, this characterization is not really sufficient if we are interested in forbidden subgraphs. In the following section we give another characterization through forbidden subgraphs which we call *flowers*.

3 A characterization of R/B-Split graphs by subgraphs

To claim the main theorem of this section, we need to define *flowers* and *handcuffs*:

DEFINITION 3.1. *A* handcuff *is an even (alternating) cycle C that can be divided into subwalks $C = C_1 + P + C_2 + P^-$, where C_1 and C_2 are odd circuits, P is a path linking the bases of C_1 and C_2, and P^- is the reverse path of P.*

It can easily been observed that handcuffs can never exist in an R/B-split graph. On the other hand, Theorem 2.1 proves that any non-R/B-split graph contains a handcuff. Hence, R/B-split graphs are characterized by forbidden handcuffs. However, we don't know how the two circuits C_1 and C_2 of a given handcuff intersect. In order to characterize R/B-split graphs by certain subgraphs, we therefore define *flowers* as subgraphs induced by special handcuffs:

DEFINITION 3.2. *A* flower *is a subgraph induced by a handcuff $C = C_1 + P + C_2 + P^-$ such that C_1 and C_2 intersect in intervals located as described in Figure 3.*

In the second flower of Figure 3 the odd cycle C_2 is the dashed walk. Note that in a general situation C_1 and C_2 could intersect at arbitrarily many intervals, and some of the intervals can have the length of 0, that is they consist of only one node.

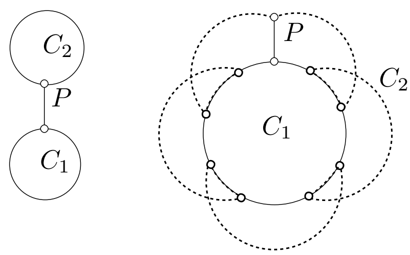

Figure 3: Flowers with one and 4 intersecting intervals.

THEOREM 3.1. *G is an R/B-Split graph if and only if there exist no flowers in G.*

In the rest of this section we will show the proof of Theorem 3.1.

For a technical reason, we always imagine a cycle to be drawn as a circuit on the plane, where a node can appear many times on the circuit. By that way we have the following definitions:

DEFINITION 3.3. *Let C be a cycle of even size. Assume that a node p occurs twice in C. C can be cut along an imaginary cutline c_p into two walks with the same starting and ending node p. As C is even, these two subcycles must be either both even or both odd. Accordingly we talk about an* even cut *or an* odd cut *c_p. A* forbidden cycle *is an even cycle with at least one odd cut. Let C be any forbidden cycle in $G = (V, R \cup B)$ and u and v two nodes each occuring more than once in C. In case the two imaginary cutlines c_u and c_v cross* inside *C, we say that c_u and c_v* cross*. Otherwise we say that c_u and c_v lie* parallel *in C (see Figure 4). If c_u and c_v are paralell odd cuts, we call the two subwalks $P_1(u, v)$ and $P_2(v, u)$ of C, which lie between the two cutlines,* parallel intervalls.

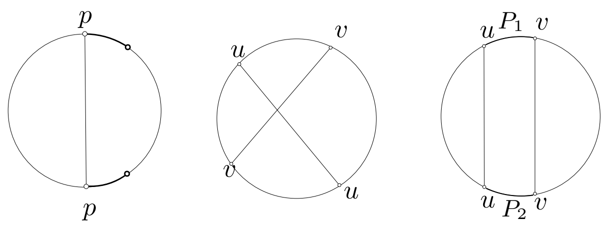

Figure 4: A forbidden cycle, two crossing cuts and two parallel odd cuts.

Observation: It's clear that we may exchange the parallel intervals P_1, P_2 or replace one by the other to get a different forbidden cycle. Two odd cycles of different color on the same base form a forbidden cycle and vice versa, thus Theorem 2.1 can be stated as follows:

THEOREM 3.2. *G is an R/B-split graph if and only if there exist no forbidden cycles.*

Flowers can be defined by a different term: normalized forbidden cycles:

DEFINITION 3.4. *A* normalized forbidden cycle *is a forbidden cycle without any even cut and having an additional property: any two parallel intervals are identical and their inner nodes do not appear elsewhere outside these parallel intervals.*

It's not hard to see that Definition 3.4 and Definition **??** are identical: Since in a normalized forbidden cycle C any two parallel intervals are identical, we may call the (alternating) paths corresponding to these pairs of identical parallel intervals *double paths* (note that it's possible that this path consists of a single node), and the remaining (alternating) paths of C as *single paths.* Consider now the subgraphs corresponding to normalized forbidden cycles. Now, a flower corresponds to a normalized forbidden cycle with more than one double paths. See Figure 5 for an example.

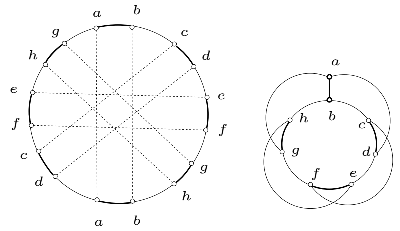

Figure 5: A normalized cycle and the corresponding flower

DEFINITION 3.5. *We call a forbidden cycle C* minimal *if there doesn't exist another forbidden cycle of smaller size in the subgraph consisting of the edges of C.*

Given a minimal forbidden cycle, we observe the following properties:

LEMMA 3.1. *In any minimal forbidden cycle an even cut can never be parallel to an odd cut. And as a consequence, in any minimal forbidden cycle each node occurs at most twice.*

Proof. Suppose c_u is an even cut parallel to an odd cut c_p in a minimal forbidden cycle C. Then by cutting off the subcycle defined by the even cut c_u which doesn't contain c_p we get a smaller even cycle containing an odd cut c_p. Thus, we get a smaller forbidden cycle, contradicting the assumption that the forbidden cycle was minimal. (Figure 6)

Now, suppose a node u occurs three times in C. Then there exist three parallel u-cuts in C. In case one of these cuts is odd, exactly one of the two other cuts has to be even. Thus we have an odd and an even parallel cut. In case all u-cuts are even, consider any odd cut c_p in C. Then at most two u-cuts cross c_p and at least one u-cut lies parallel to c_p. In both cases we get a situation where an odd cut lies parallel to an even cut, due to the observation above, C couldn't be a minimal forbidden cycle.

LEMMA 3.2. *For any pair of parallel intervalls $P_1(u, p)$ and $P_2(p, u)$ of a minimal forbidden cycle C holds: Each*

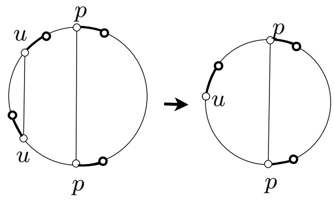

Figure 6: The forbidden cycle C and the shorter one C'.

interval is simple and their inner nodes do not appear elsewhere outside P_1 and P_2.

Proof. First we have $|P_1(u,p)| = |P_2(u,p)|$ because otherwise we could replace the bigger interval by the smaller one toget a smaller forbidden cycle. Now by replacing one interval by the other we will get another forbidden cycle of the same size.

In case a node occurs twice in one of these intervals, we could replace the other one by this to get a forbidden cycle where there is a node appearing four times. Similarly if an inner node of P_1 appears outside P_1, P_2, we could replace P_2 by P_1 to get a forbidden cycle with a node appearing more than twice.

In both cases we get into a contradiction according to Lemma 3.1.

Now we prove that any non-R/B-split graph contains a minimal forbidden cycle without even cuts.

LEMMA 3.3. *If G is not an R/B-split graph, there exists a minimal forbidden cycle without even cuts.*

Proof. Let C be any minimal forbidden cycle with an odd cut c_p. We write $C = C_1(p) + C_2(p)$ where $C_i(p)$ are the odd cycles defined by the cut line c_p. Consider the cycle $\tilde{C} = C_1(p) + C_2(p)^-$ obtained by reversing the order of the odd cycle C_2 on C. Then $\tilde{C}$ is also a minimal forbidden cycle with odd cut c_p. For any cut c_u of C, the property to parallel or cross c_p keeps the same in $\tilde{C}$. Moreover, any cut parallel to c_p in C has the same parity in C as it has in $\tilde{C}$. But any cut crossing c_p has a different parity in C as it has in $\tilde{C}$.

Let C be a minimal forbidden cycle in G with an odd cut c_p and an even cut c_u. In case c_p is the only odd cut, we know from Lemma 3.1 that all additional even cuts cross c_p. Now all cuts in $\tilde{C} = C_1(p) + C_2(p)^-$ are odd. In case there exists a second odd cut c_q in C, we know from Lemma 3.1 that the even cut c_u crosses c_p as well as c_q. We claim that c_p and c_q lie parallel in C, as otherwise c_q would be an even cut parallel to the odd cut c_u in $\tilde{C} = C_1(p) + C_2(p)^-$, which is a contradiction to Lemma 3.1. Thus, every odd cut lies parallel to c_p and hence all cuts in $\tilde{C}$ are odd.

Proof. (**Theorem 3.1**) We are now ready to prove the main theorem. Due to Theorem 3.2 the graph is not an R/B-split graph iff it contains a forbidden cycle. Take a minimal forbidden cycle without even cuts, due to Lemma 3.3 we can do this. Let $(P_1, P_1'), (P_2, P_2')..,(P_k, P_k')$ be the pairs of maximal parallel intervals. Applying Lemma 3.2 we know that they are simple and node disjoint with the rest of the cycle. Replacing P_i' by P_i we get a normalized cycle whose induced subgraph is a flower.

Note that the above proof also gives an algorithm for finding a normalized forbidden cycle.

4 A characterization of Kőnig-Egerváry Graphs

The MINIMUM VERTEX COVERING PROBLEM, is well known to be $\mathcal{NP}$-complete [7]. Dually the MAXIMUM MATCHING PROBLEM is in $\mathcal{P}$. Obviously the weak duality $\tau(G_B) \geq \nu(G_B)$ holds for every graph G_B. A graph G_B is a *Kőnig-Egerváry Graph (KEG)* in case $\tau(G_B) = \nu(G_B)$.

Using the result in the previous section, we give a subgraph excluded characterization of KEGs.

We first prove that the problem to decide whether a graph with a perfect matching is a KEG, can be easily reduced to an R/B-SPLIT PROBLEM:

LEMMA 4.1. *Let $G_B = (V, B)$ be an undirected graph with a perfect matching R. Then G_B is a KEG iff $G = (V, R \cup B)$ is an R/B-split graph.*

Proof. Given a perfect matching R of $G_B = (V, B)$, a subset of vertices V_R is a vertex cover of size $|R|$ iff V_R is stable in $G_R = (V, R)$ and V_B is stable in G_B.

It turns out that when the maximum matching R is not a perfect matching, then by replacing all the uncovered nodes (the nodes which don't meet R) by a new red-edge, and connect the endnodes of this edge to the nodes that the original node was adjacent to, we get back to the case when the red edge set is a perfect matching. More precisely, we have the following lemma:

LEMMA 4.2. *Let $G_B = (V, B)$ be an undirected graph with a maximum matching R. Let V_0 denote the node set which are not covered by R. Replace every node $v \in V_0$ by a new edge: $r_v = (v', v")$, and any edge (v, x) by a pair of new edges (v', x) and $(v", x)$. Call the resulting graph $G'_{B'} = (V', B')$. Let $R' = R \cup_{v \in V_0} r_v$. Then R' is a perfect matching in $G'_{B'}$ and G_B is a KEG iff $G' = (V', R' \cup B')$ is an R/B-split graph.*

Proof. It's clear that R' is a perfect matching of $G'_{B'}$. Let $C \subset V$ be a minimum cover of B, and let's assume that $|C| = |R|$ then clearly $C \cap V_0 = \emptyset$. Extend C to a cover in $G'_{B'}$ by adding $\cup_{v \in V_0} v'$. This extension gives a cover of size $|R'|$. Hence $\nu(G_B) = \tau(G_B)$ implies $\nu(G'_{B'}) = \tau(G'_{B'})$.

Conversely, assume that there exists a cover $C' \subset V'$ such that $|C'| = |R'|$. It's easy to see that $C' \cap V$ is a cover of B of size $|R|$. By this, we proved that G_B is a KEG iff $G'_{B'}$ is a KEG. Using Lemma 4.1 we finished the proof.

We need the following definition to claim the main theorems in this section:

DEFINITION 4.1. *Let R be a perfect matching of $G_B = (V, B)$. An* elementary even subdivision *of an edge $(u, v) \in B$ is the replacement of (u, v) by the path of edges $((u, x), (x, y), (y, v))$ and R by*

$$\hat{R} = \begin{cases} (R \setminus (u,v)) \cup \{(u,x),(y,v)\} & \text{if } (u,v) \in R \\ R \cup (x,y) & \text{otherwise.} \end{cases}$$

An even subdivision *of an edge is the result of a sequence of elementary even subdivisions of an edge.*

THEOREM 4.1. *Let R be a perfect matching of $G_B = (V, B)$. Then either G_B is a KEG or there exists a subgraph $H = (\hat{V}, \hat{B})$ with a perfect matching $\hat{B} \cap R$ such that H is the result of even subdivisions of some edges of one of the configurations in picture 7:*

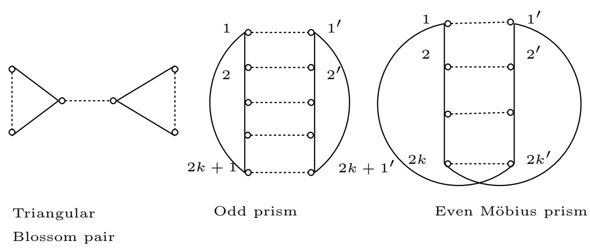

Figure 7: The dashed edges are the matching edges.

THEOREM 4.2. *Let R be a maximum matching of $G_B = (V, B)$. Then either G_B is a KEG or there exists a subgraph $H = (\hat{V}, \hat{B})$ of G such that H is the result of even subdivisions of some edges of one of the configurations in picture 7 and in picture 8:*

Proof. (**Theorem 4.1**) Let H be a flower in $G = (V, R \cup B)$. It is easy to see that no node of H is incident to four edges in H (i.e. there exists no double interval that consists of a single node in H), as otherwise two matching-edges would have to be adjacent. Moreover

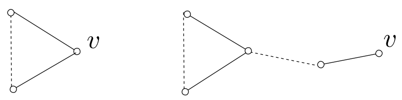

Figure 8: v is a uncovered (exposed) node

as nodes of degree three in H correspond to end nodes of double intervals, double intervals have to be odd alternating paths with matching edges at the end. For the same reason single intervals have to be odd alternating paths with non-matching edges at the end. The converse implications are obvious.

Proof. (**Theorem 4.2**) If R is a perfect matching, then we are done. Consider the graph G' as in Lemma 4.2, G_B is KEG iff $G' = (V', R' \cup B')$ is an R/B-split graph. Let H be a flower in G'. If H doesn't contain any exposed edge, i.e an edge that is a result of expansion of an uncovered vertex in $G = (V, R \cup B)$, then we are done.

We claim that H cannot contain more than one exposed edge. It's true because otherwise one can find an alternating path in G between two uncovered nodes, which is a contradiction since we assumed that R was a maximum matching.

Now assume H contains an exposed edge ($v'v$"). Consider H as a normalized forbidden cycle with some parallel intervals. Among them take an interval $P(a, b)$ such that either v' is contained in $P(a, b)$ or P(a,b) is the closest parallel interval to v' in clockwise order. Consider the latter case see Figure 9, (the other one is similar). It's easy to see that the curve ($v'a$) drawn on the Figure is simple and it can be obtained from the figurations in Figure 8 under the even subdivision operations.

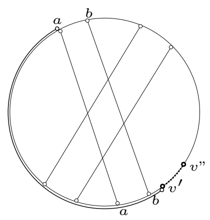

Figure 9: H contains an exposed edge $v'v$"

Conversely, if G has a subgraph H such that H is the result of even subdivisions of some edges of one of the configurations in picture 8: then in G' one can easily find forbidden cycles, which implies that G' is not an R/B-split graph.

By this and using Lemma 4.1, we finished the proof.

5 Additional results

We now show how theorems of Deming-Sterboul, Lovász and Földes-Hammer can be derived from our results in previous sections.

Deming [2] and Sterboul [16] in 1979 independently proved a characterization of KEGs, but their characterizations are in terms of some forbidden configurations which are not simple subgraphs. They defined the followings: Given an undirected graph $G = (V, B)$ and a maximum matching R, a *blossom relative* to R with the base p is a blue odd simple subcycle with the base p. *Exposed* nodes are the nodes not covered by R. The *sterm* is an even length alternating path joining the base of a blossom and an exposed node for R. A *flower*, which is different from our definition of flower, is a blossom and its sterm. A *posy* or a *blossom pair* consists of two not necessarily disjoint blossoms joined by an odd length alternating path whose first and last edges belong to R, the two end nodes are exactly the bases of the two blossoms.

Corollary 5.1. (Deming-Sterboul [2], [16]) *Let G be an undirected graph with no loops or multiple edges and let M be a maximum matching in G. G is a KEG iff there exist neither flowers nor posies relative to M*

Proof. The only thing we need to show is that odd prisms and even möbius prisms in Figure 7 are posies. The other cases are trivial. One can easily see that $(1, 1')$ is the odd length path, and $1, 2, 2', 3', 3...2k + 1', 2k + 1, 1$ and $1', 2', 2...2k + 1, 2k + 1', 1'$ are two blossoms for the case of odd prisms and $1, 1', 2', 2, 3, 3'...2k - 1, 2k - 1', 2k', 1$ and $1', 1, 2, 2'...2k - 1', 2k - 1, 2k, 1'$ are two blossoms for the case of even m ́obius prisms.

Lovász [13] used the theory of matching cover graph to give a subgraphs excluded characterization for KEGs that contain a *perfect* matching. However, his characterization is based on a *particular* perfect matching, not on an arbitrary given one. The following Corollary is an extended version of Lovász's theorem for general graphs.

Corollary 5.2. (Lovász [13]) *Let G be an undirected graph with no loops or multiple edges. G is not a KEG iff there* exists *a maximum matching M in G and a subgraph H of G such that $(H, M \cup E(H))$ is an even subdivision of one of the configurations in Figure 10.*

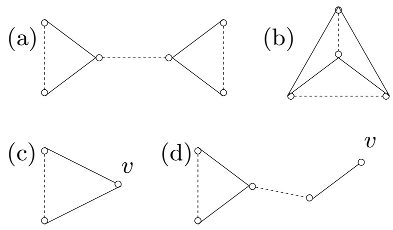

Figure 10: Dashed edges are the matching edges, v is an exposed node.

Proof. The difference between Lovász's theorem and ours is our theorem works for any given maximum matching while in Lovász's theorem one has to find one particular maximum matching. So we start with an arbitrary maximum matching M'. Using Theorem 4.2, we get a subgraph H' which is one of the forbidden configurations in Figure 7 and in the Figure 8. We will modify M' to another maximum matching M to get the configurations in Figure 10.

We are done as soon as H is one of the configurations of Figure 8. If H' is an odd prism of Figure 7, interchange matching edges and non-matching edges in the following collection of disjoint cycles: $(2, 2', 3', 3); (4, 4', 5', 5); ...; (2k, 2k', 2k + 1', 2k + 1)$. It's easy to see that $|M| = |M'|$, thus $|M|$ is a maximum matching also. Now delete $(2, 2'); (3, 3'); ..; (2k + 1, 2k + 1')$ we get a subgraph which is a configuration of Figure 10(a). A similar operation can be applied when H' is an even möbius prism of Figure 7 : interchange matching and non matching edges of $(3, 3', 4', 4); (5, 5', 6', 6); ...; (2k - 1, 2k - 1', 2k', 2k)$ and delete $(3, 3'); ..; (2k+1, 2k+1')$ we get the configuration Figure 10 (b).

Recall that a subset $X \subseteq V$ is a *clique* in the graph $G_B = (V, B)$, if any two members of X are joined by an edge of B. Thus each clique in G_B is a stable set in the complement graph $\bar{G}_B = (V, \bar{B})$. If the red graph $G_R = (V, R)$ of $G = (V, R \cup B)$ is the complement graph of the blue graph G_B (i.e. if $R = \bar{B}$), we therefore find that G is an R/B-Split graph if and only if the blue graph G_B as well as the red graph G_R themselves can be split into a clique and a stable set each. Thus G is an R/B-Split graph, if and only if G_B and G_R are *Split graphs.*

Corollary 5.3. (Földes-Hammer [5]) *A graph is a Split graph if and only if it contains no induced subgraph isomorphic to $2K_2, C_4$ or C_5.*

This theorem can be proved by showing the Red-Blue graph $G = (V, R \cup B)$ with $R = \bar{B}$ is an R/B-Split graph if and only if G contains no subgraph isomorphic to one of the three types shown in Figure 11. The proof is not difficult (appearing in the full version of this paper). Basically one can take a minimal normalized forbidden cycle, using the fact that there is either a red

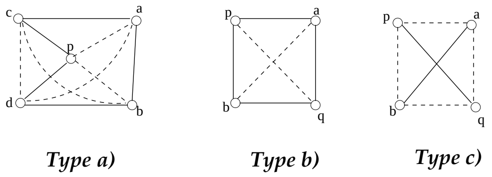

Figure 11: The three induced forbidden subgraphs in case $R = \bar{B}$.

or a blue edge between any pair of nodes and it can be seen that indeed the minimal normalized forbidden cycles can only be one of the configurations in Figure 11.

Weighted R/B-split graphs: Another application of R/B-split graphs is the characterization of integral solutions of certain inequalities' system. Here we define the problem and claim the result, detailed proofs and connections with Schrijver's theorem are discussed in the full version of this paper.

Given a Red-Blue graph $G = (V, R \cup B)$ and a weighting $b : R \cup B \to \mathbb{Z}$ on the edgeset. In the WEIGHTED R/B-SPLIT PROBLEM, we search for a vector $x : V \to \mathbb{Z}$, such that x satisfies for each edge $(i,j) \in R \cup B$:

$$\begin{aligned} x_i + x_j &\leq b(e) \text{ if } e = (i,j), e \in R \text{ and} \\ -x_i - x_j &\leq b(e) \text{ if } e = (i,j), e \in B \end{aligned}$$

Note that the Weighted R/B-Split problem reduces to the ordinary R/B-Split problem in the special case, where each red edge has weight 1 and each blue edge has weight -1.

The weight of an alternating walk W is defined as $b(W) = \sum_{e \in W} b(e)$. W is *tight* if $b(W) = 0$. Recall Definition 3.3: A forbidden cycle is an even cycle with at least one odd cut. Let W be a *tight forbidden cycle*. Using the odd cut, W can be decomposed into two odd cycles W_1 and W_2. The *parity* of W is defined as $parity(W) = b(W_1) \equiv b(W_2) \pmod 2$. Let us denote a tight forbidden cycle *odd* if its parity is 1. It can be showed that this definition is well defined in case the decomposition is not unique.

THEOREM 5.1. *The Weighted R/B-Split Problem of the graph (G, b) is solvable, if and only if no negative weighted even cycle and no tight odd flower exists in (G, b).*

6 Conclusion and future work

We gave a model of R/B-split graphs which generalizes KEGs and Split graphs and proved a characterization: a graph is R/B-split iff it doesn't contain one of the forbidden subgraphs called flowers.

One can ask: Given $G = (V, R \cup B)$ what is the maximum R/B-split subgraph? There are more than one version of this question:

1. What is the maximum number of nodes that can be partitioned into a red and a blue stable set? U. Faigle et al [4] gave a partial answer: the problem is $\mathcal{NP}$-hard even for comparability graphs, and there can be no approximation ratio better than $\frac{31}{32}$ unless $\mathcal{P} = \mathcal{NP}$. One of the questions is what is the lower bound of the approximation ratio?
2. One can also ask what is the minimum number of edges that we can erase such that the resulting graph is R/B-split?
3. How about the case when one's allowed to erase the red edges only. What is the minimum number of red edges that we can erase such that the resulting graph is R/B split? We remark that this problem is not easier than the Minimum Vertex Cover problem since if the graph has a perfect matching and the red edges are edges in a perfect matching, then this question in fact asks the difference of the minimum vertex cover and the maximum matching.

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